Analysis of TEM Mode on a Curved Coaxial Transmission Line

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Abstract —Through the use of a perturbation approach, explicit expressions are derived for the changes in the electromagnetic field structure that occur when a TEM mode on a coaxial transmission line enters a bend in the line. All of these changes are evaluated to at least first order in the inverse of the radius of curvature of the coaxial line. An explicit expression is also constructed for the first nonvanishing correction term to the propagation constant, which turns out to be of second order. Graphical results are presented for the variation of the propagation constant as a function of curvature and other parameters characteristic of the coaxial line.

I. Introduction

TWO OF THE earliest authors to employ a perturbation technique to analyze propagation of electromagnetic waves in curved structures were Jouguet [1] and Riess [2]. In their papers, Maxwell's equations expressed in conveniently selected curved coordinates were expanded in inverse powers of the radius of curvature. From the resulting equations, first-order solutions for the propagation constant and field distribution were obtained for waveguides of rectangular and circular cross sections. In particular, it was found that the TE_{01} mode in a circular waveguide breaks up into the sum of two distinct modes, which resemble the sum and difference of the TE_{01} and TM_{11} modes, as it propagates around a bend. By contrast, the TE_{01} mode in a rectangular waveguide propagates with little alteration in structure around a bend. By employing concepts from the theory of coupled transmission lines, Albersheim [3] was able to derive further results relating to the propagation of TE_{01} waves in curved waveguides. In so doing, he reproduced some of the equations deduced by Jouguet.

Propagation of electromagnetic energy in curved circular waveguides was later addressed again by means of a perturbation technique, this time by Lewin [4], [5]. In a rather elegant fashion, he utilized bicomplex variables to decouple the various field components from Maxwell's equations, thereby creating a decoupled wave equation for a new field combination. This field combination was essentially a linear combination of the components of the electric and magnetic fields taken along the continuously bending axis of the waveguide. In constructing the decoupled wave equation, a curvilinear set of local orthogonal coordinates erected around the axis of the bent waveguide was employed. This set of coordinates was developed from the work of Tang [6], who had shown how orthogonal systems could be constructed from the Serret–Frenet frame of an arbitrary curve. The decoupled wave equation was then solved to first order by the previously employed perturbation technique of expanding in inverse powers of the radius of curvature. As a result, expressions for the axial fields and propagation constant characterizing each of the two distinct stable combinations of the TE_{01} and TM_{11} modes were obtained.

Perturbation approaches have also been used to analyze wave propagation in curved structures other than waveguides of constant cross section and curvature. In particular, Lewin [7] applied a perturbation technique to solve the problem of a rectangular waveguide uniformly twisted about a straight axis. In addition, Chang and Kuester [8] analyzed bent dielectric guides of arbitrary cross section, and Tripathi and Wolff [9] treated curved microstrip resonators, both using perturbation approaches. Although many other authors have dealt with the problem of electromagnetic waves on various curved guiding structures, it appears that a detailed analysis of a propagating TEM mode on a bent coaxial transmission line has not yet been performed. Specifically, explicit analytical expressions for the propagation constant and field structure present in a bent coaxial line have not yet been deduced. It is the intent of this paper, then, to develop such analytical expressions by means of the perturbation technique first employed by Jouguet and Riess.

With regard to the organization of this paper, it commences with a brief presentation of the characteristics of the pure TEM mode supported by a straight coaxial transmission line. An appropriate orthogonal coordinate system for the bent coaxial line is then established, after which Maxwell's wave equation is written in terms of these coordinates and, subsequently, expanded in inverse powers of the radius of curvature. This expansion permits analytical forms for the axial fields to be derived to first order and, later, to second order. Lewin's wave equation for his field combination of axial components [5] is then used in conjunction with the derived first-order axial field expressions to show that the propagation constant remains unaltered to first order. For the sake of completeness, the field components transverse to the axis of the bent coaxial line are also evaluated to first order by

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employing the appropriate Maxwell's equations. In an effort to obtain the first nonvanishing correction term in the propagation constant, the axial field components are then evaluated to second order in the inverse of the radius of curvature. Taken in conjunction with Lewin's wave equation, these results are utilized to obtain an expression for the second-order correction to the propagation constant, which is the desired first nonvanishing correction term. Finally, graphical results depicting the behavior of the propagation constant in a bent coaxial line are presented.

II. First-Order Axial Fields

For a pure TEM mode propagating on a coaxial transmission line, the electric field has only a radial component and the magnetic field only an azimuthal component. This simple mode structure, which propagates unattenuated, is realized in a straight section of coaxial line having conductors of infinite conductivity. The fields existing between the inner and outer conductors in the TEM mode are expressible as

$$E_{\rho 0}=\sqrt{\frac{\mu}{\epsilon}}\,\frac{I}{2\pi\rho} \qquad H_{\phi 0}=\frac{I}{2\pi\rho}. \tag{1}$$

The variables ρ and ϕ denote, respectively, the radial and azimuthal cylindrical coordinates, while I denotes the axial (z-directed) current flowing in the inner conductor. In these equations, the multiplicative propagating factor $\exp(j\omega t - jkz)$ has been suppressed. Here ω denotes the angular frequency and k the wavenumber or straight-line propagation constant. In terms of the permittivity ϵ and the permeability μ of the dielectric between the inner and outer conductors, k and ω are related by $k^2=\omega^2\mu\epsilon$.

To solve for the propagating fields in a bent section of coaxial transmission line, it is desirable to establish an orthogonal coordinate system erected around the coaxial line axis. Following the work of Tang [6] and Lewin [4], we find that it is convenient to employ the cylindrical orthogonal set of coordinates shown in Fig. 1. Here the location of an arbitrary point P is seen to be specified by the coordinate set (ρ,ϕ,s), where ρ and ϕ constitute transverse polar coordinates relative to the continuously bending axis of the coaxial line. The axial coordinate s specifies the location of the cross-sectional plane containing P along the transmission line axis relative to some reference point. Assuming that the coaxial line is bent in one plane and that it has a constant radius of curvature R, we can write the metric coefficients for this coordinate system as

$$h_\rho=1 \qquad h_\phi=\rho \qquad h_s=1-\frac{\rho}{R}\cos\phi. \tag{2}$$

It is to be noted that these metric coefficients reduce to those appropriate for the normal cylindrical coordinate system as the coaxial line is straightened, i.e., as $R\to\infty$.

Having established a convenient orthogonal set of coordinates for the problem at hand, we commence now to solve Maxwell's electromagnetic equations for the propagating field distribution. For an assumed $e^{j\omega t}$ time dependence, Maxwell's three-dimensional wave equation for the magnetic field is

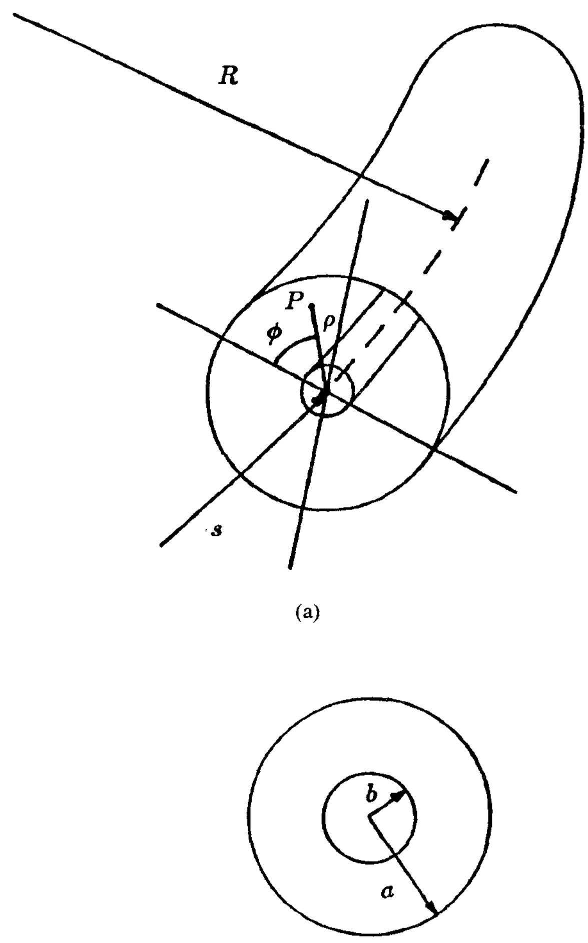

Fig. 1. Geometry of curved coaxial transmission line. (a) Definition of orthogonal coordinate system employed in the analysis. (b) Cross-sectional view of line.

$$-\nabla^2\boldsymbol{H}=\nabla\times\nabla\times\boldsymbol{H}-\nabla(\nabla\cdot\boldsymbol{H})=k^2\boldsymbol{H} \tag{3}$$

where $\boldsymbol{H}=H_\rho\hat{\rho}+H_\phi\hat{\phi}+H_s\hat{s}$. The s component of this equation yields a partial differential equation for H_s, the magnetic field component in the direction of propagation. Since we seek a solution propagating as $e^{-\gamma s}$, the differential operator $\partial/\partial s$ can everywhere be replaced by $-\gamma$, whereupon the field components will depend only upon ρ and ϕ. Upon making this replacement and employing the metric coefficients given in (2), we find that the differential equation for H_s becomes

$$\frac{-2\gamma}{R}\cos\phi H_\rho+\frac{2\gamma}{R}\sin\phi H_\phi+\frac{h_s^2}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{H_s}{h_s}\cos\phi\right)-h_s^2\nabla_t^2H_s-\frac{h_s^2}{R\rho}\frac{\partial}{\partial\phi}\left(\frac{H_s}{h_s}\sin\phi\right)=\left(\gamma^2+h_s^2k^2\right)H_s. \tag{4}$$

Here ∇_t^2 denotes the so-called transverse Laplacian operator, i.e.,

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}. \qquad (5)$$

Equation (4) is a complicated partial differential equation into which the solutions (as yet unknown) for the transverse field components H_ϕ and H_ρ are coupled. Written as such, it cannot be solved in closed form.

A solution for this equation can be effected if one employs a perturbation technique [4], whereby the field components and the propagation constant are each expanded into a series containing inverse powers of the radius of curvature, i.e.,

$$H_s = H_{s0} + \frac{H_{s1}}{R} + \cdots \qquad H_\rho = H_{\rho 0} + \frac{H_{\rho 1}}{R} + \cdots$$
$$H_\phi = H_{\phi 0} + \frac{H_{\phi 1}}{R} + \cdots \qquad \gamma^2 = \gamma_0^2\left(1 + \frac{A_1}{R} + \cdots\right). \qquad (6)$$

The zeroth-order terms appearing here are those which are appropriate for the TEM mode in a straight coaxial transmission line. Accordingly, we have

$$H_{s0} = H_{\rho 0} = 0 \qquad \gamma_0 = jk \qquad (7)$$

and $H_{\phi 0}$ is given by the second expression of equations (1). To first order in R^{-1}, (4) reduces, with the aid of (1), (2), and (6), to

$$\nabla_t^2 H_{s1} = \frac{jkI\sin\phi}{\pi\rho}. \qquad (8)$$

The solution to this differential equation has been found, following the general procedure outlined in Andrews [10], to be

$$H_{s1}(\rho,\phi) = \frac{jkI\sin\phi}{2\pi(b^2-a^2)}\left[\rho\{b^2\ln(\rho/b) - a^2\ln(\rho/a) + a^2 - b^2\} + \frac{a^2b^2}{\rho}\ln(a/b)\right]. \qquad (9)$$

As shown in Fig. 1, b constitutes the radius of the inner conductor and a the inner radius of the outer conductor. The boundary conditions utilized in arriving at (9) are specified by the vanishing of $\partial H_{s1}/\partial\rho$ at $\rho = a$ and $\rho = b$.

Through the use of a similar procedure, a solution for the axial electric field to first order in R^{-1} can be obtained. Since the three-dimensional wave equation is identical in form to that for the magnetic field, i.e., $-\nabla^2\boldsymbol{E} = k^2\boldsymbol{E}$, the differential equation for E_s is deducible from (4) simply by replacing H wherever it appears with E. A solution to the resulting equation can then be effected by employing the perturbation technique discussed above, whereupon the field components can be expanded as

$$E_s = E_{s0} + \frac{E_{s1}}{R} + \cdots$$
$$E_\rho = E_{\rho 0} + \frac{E_{\rho 1}}{R} + \cdots$$
$$E_\phi = E_{\phi 0} + \frac{E_{\phi 1}}{R} + \cdots. \qquad (10)$$

Recognizing that $E_{s0} = E_{\phi 0} = 0$ and that $E_{\rho 0}$ is given by the first expression in (1), we find that, to first order in R^{-1}, the differential equation for E_{s1} reduces to

$$\nabla_t^2 E_{s1} = -\sqrt{\frac{\mu}{\epsilon}}\,\frac{jkI\cos\phi}{\pi\rho}. \qquad (11)$$

The solution to this differential equation that satisfies the perfect conductor boundary conditions that E_{s1} must vanish at $\rho = a$ and $\rho = b$ is given by

$$E_{s1}(\rho,\phi) = \sqrt{\frac{\mu}{\epsilon}}\,\frac{jkI\cos\phi}{2\pi(b^2-a^2)}\left[\rho\{a^2\ln(\rho/a) - b^2\ln(\rho/b)\} + \frac{a^2b^2}{\rho}\ln(a/b)\right]. \qquad (12)$$

With the construction of the equations for H_{s1} and E_{s1}, we have obtained the complete axial solutions to order R^{-1} for the propagating field configuration in a bent section of coaxial line.

III. First-Order Correction to Propagation Constant

To find the first-order correction to the propagation constant γ, it is necessary to resort to the decoupled wave equation developed in [4] for the axial field combination $G_\pm$, defined as

$$G_\pm = E_s \pm j\sqrt{\frac{\mu}{\epsilon}}\,H_s. \qquad (13)$$

The wave equation satisfied by $G_\pm$ is

$$\begin{aligned}\nabla_t^2 G_\pm &+ (k^2 + \gamma^2/h_s^2)G_\pm \\ &= \frac{1}{R\Gamma^2}\left\{\frac{1}{R}(k^2 - \gamma^2/h_s^2)G_\pm + h_s(k^2 + 3\gamma^2/h_s^2)\right. \\ &\quad\cdot\left[\cos\phi\frac{\partial G_\pm}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial G_\pm}{\partial\phi}\right] \\ &\quad\left.\pm 2\gamma k\left[\sin\phi\frac{\partial G_\pm}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial G_\pm}{\partial\phi}\right]\right\} \qquad (14)\end{aligned}$$

where $\Gamma^2 = k^2h_s^2 + \gamma^2$. Now $G_\pm$ is itself expandable in inverse powers of R according to

$$G_\pm = G_{\pm,0} + \frac{G_{\pm,1}}{R} + \cdots \qquad (15)$$

where $G_{\pm,0} = 0$ since $E_{s0} = H_{s0} = 0$ and

$$G_{\pm,1} = E_{s1} \pm j\sqrt{\frac{\mu}{\epsilon}}\,H_{s1}. \qquad (16)$$

By retaining terms only to first order in R^{-1}, we can construct the differential equation governing $G_{\pm,1}$ from (14); the result is

$$\begin{aligned}(-2\rho\cos\phi - A_1)\nabla_t^2 G_{\pm,1} &= -2\left[\cos\phi\frac{\partial G_{\pm,1}}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial G_{\pm,1}}{\partial\phi}\right]\\ &\pm 2j\left[\sin\phi\frac{\partial G_{\pm,1}}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial G_{\pm,1}}{\partial\phi}\right]. \end{aligned}\quad (17)$$

Here A_1, as shown by the last expression in (6), is the coefficient of the first-order term in the expansion of γ^2 in inverse powers of the radius of curvature. Upon substituting the relationships for H_{s1} and E_{s1} derived above into (17), we find that this equation is satisfied only if

$$A_1 = 0. \quad (18)$$

Consequently, $\gamma^2 = \gamma_0^2 = -k^2$ through first order in R^{-1}, which is to say that the propagation constant for a straight section of line remains unaltered through first order in R^{-1} in a bent section of line. This corresponds to a specific finding in [5], namely, that there is no first-order change in the propagation constant in a circular waveguide when the TM_{11} mode with electric polarization parallel to the plane of the bend propagates into a curved section of guide. We proceed now to evaluate the first-order terms for the transverse components of the field distribution present in a bent coaxial line.

IV. First-Order Transverse Fields

The partial differential equation governing $H_{\rho 1}$, the first-order correction to H_ρ, can be obtained by equating the ρ components on both sides of (3). After replacing $\partial/\partial s$ by $-\gamma$ and utilizing the metric coefficients given by (2) in conjunction with the indicated differentiations, we obtain the following differential equation for $H_{\rho 1}$, valid to first order in R^{-1}:

$$\frac{2}{\rho^2}\frac{\partial H_{\phi 1}}{\partial\phi} + \frac{\sin\phi}{\rho}H_{\phi 0} - \nabla_t^2 H_{\rho 1} + \frac{H_{\rho 1}}{\rho^2} = 0. \quad (19)$$

Although the analytical form for $H_{\phi 0}$ is given by the second expression in (1), the analytical form for $H_{\phi 1}$ is as yet unknown. Nevertheless, $H_{\phi 1}$ can be eliminated from (19) through the use of the Maxwellian equation

$$\nabla\cdot\boldsymbol{H} = 0 = \frac{1}{\rho h_s}\left[\frac{\partial}{\partial\rho}(\rho h_s H_\rho) + \frac{\partial}{\partial\phi}(h_s H_\phi) + \frac{\partial}{\partial s}(\rho H_s)\right] \quad (20)$$

which, to first order in R^{-1}, becomes

$$0 = \frac{\partial}{\partial\rho}(\rho H_{\rho 1}) + \rho\sin\phi H_{\phi 0} + \frac{\partial H_{\phi 1}}{\partial\phi} - \gamma_0\rho H_{s1}. \quad (21)$$

Combining (19) and (21) to eliminate $H_{\phi 1}$, we obtain the desired differential equation for $H_{\rho 1}$, i.e.,

$$\nabla_t^2 H_{\rho 1} + \frac{H_{\rho 1}}{\rho^2} + \frac{2}{\rho}\frac{\partial H_{\rho 1}}{\partial\rho} = \frac{2\gamma_0}{\rho}H_{s1} - \frac{\sin\phi}{\rho}H_{\phi 0}. \quad (22)$$

The form of the forcing function appearing on the right-hand side is known completely, being given in terms of the expressions found in (1) and (9). After performing lengthy and nontrivial manipulations, we find the solution to (22) to be

$$\begin{aligned}H_{\rho 1}(\rho,\phi) = \frac{I\sin\phi}{\pi(b^2-a^2)}\Bigg\{&\frac{k^2a^2b^2k_1}{(b^2-a^2)\rho^2} + \frac{a^2b^2}{4\rho^2}\ln(a/b)\\ &+\frac{k^2k_2}{(b^2-a^2)} - \frac{1}{4}f_1(\rho) + \frac{7}{32}k^2\rho^2(b^2-a^2)\\ &-\frac{k^2a^2b^2}{2(b^2-a^2)}\ln(a/b)f_1(\rho) - \frac{k^2\rho^2}{8}f_1(\rho)\Bigg\}\end{aligned} \quad (23)$$

where

$$\begin{aligned}k_1 &= \frac{7}{32}(b^2-a^2)^2 + \frac{a^2b^2}{2}\ln^2(a/b)\\ k_2 &= -\frac{7}{32}(a^6 - a^4b^2 - a^2b^4 + b^6)\\ &\quad + \frac{a^2b^2}{8}(b^2-a^2)\ln(a/b)\\ f_1(\rho) &= b^2\ln(\rho/b) - a^2\ln(\rho/a).\end{aligned} \quad (24)$$

The solution for $H_{\rho 1}$ conforms to the boundary condition which requires that the normal component of the magnetic field vanish at the surface of a perfect conductor. In terms of the geometry under investigation, this implies that $H_{\rho 1}$ must vanish at $\rho = a$ and $\rho = b$.

The remaining first-order component of the magnetic field, $H_{\phi 1}$, can be found from the differential equation which results from equating the ϕ components on both sides of (3). It is, however, easier to obtain a solution for this quantity by integrating (21) with respect to ϕ since $H_{\rho 1}$ is now known. Upon performing the integration, we find that $H_{\phi 1}$ is expressible as

$$\begin{aligned}H_{\phi 1}(\rho,\phi) = \frac{I\cos\phi}{\pi(b^2-a^2)}\Bigg\{&-\frac{k^2a^2b^2k_1}{(b^2-a^2)\rho^2} - \frac{a^2b^2}{4\rho^2}\ln(a/b)\\ &+\frac{k^2k_2}{(b^2-a^2)} - \frac{1}{4}f_1(\rho) + \frac{k^2\rho^2}{32}(b^2-a^2)\\ &-\frac{k^2a^2b^2}{2(b^2-a^2)}\ln(a/b)f_1(\rho)\\ &+\frac{k^2\rho^2}{8}f_1(\rho) + \frac{1}{4}(b^2-a^2)\Bigg\}.\end{aligned} \quad (25)$$

From this relationship, one can calculate to first order in R^{-1} the amount by which the already existing azimuthal magnetic field of the TEM mode is modified in a bend of the coaxial line.

To find first-order solutions for the transverse components of the electric field, namely $E_{\rho 1}$ and $E_{\phi 1}$, it is easiest to resort directly to Maxwell's curl equations.

From the ϕ component of the Maxwell equation $\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}$, we obtain

$$E_\rho = \frac{j\omega\mu h_s}{\gamma} H_\phi - \frac{1}{\gamma}\frac{\partial}{\partial\rho}(h_s E_s). \tag{26}$$

To first order in the inverse of the radius of curvature, this equation becomes, with the aid of the last expression in (2),

$$E_{\rho 1} = \sqrt{\frac{\mu}{\epsilon}}\,(H_{\phi 1} - \rho\cos\phi H_{\phi 0}) + \frac{j}{k}\frac{\partial E_{s1}}{\partial\rho}. \tag{27}$$

Upon utilizing (1), (12), and (25), we obtain finally the following solution for $E_{\rho 1}$:

$$\begin{aligned} E_{\rho 1}(\rho,\phi) = \sqrt{\frac{\mu}{\epsilon}}\,\frac{I\cos\phi}{\pi(b^2-a^2)}\Bigg\{ & -\frac{k^2a^2b^2k_1}{(b^2-a^2)\rho^2} \\ & + \frac{a^2b^2}{4\rho^2}\ln(a/b) + \frac{k^2k_2}{(b^2-a^2)} + \frac{1}{4}f_1(\rho) \\ & + \frac{k^2\rho^2}{32}(b^2-a^2) - \frac{k^2a^2b^2}{2(b^2-a^2)}\ln(a/b)f_1(\rho) \\ & + \frac{k^2\rho^2}{8}f_1(\rho) + \frac{1}{4}(b^2-a^2)\Bigg\}. \end{aligned} \tag{28}$$

A comparison with (23) and (25) reveals that no new terms appear in the expression for $E_{\rho 1}$ that were not present in the expressions for the first-order transverse magnetic field components.

By proceeding in a similar manner, we can deduce a solution for the first-order contribution to the azimuthal electric field in a bent section of coaxial line. Specifically, we find that the ϕ component of the Maxwell equation $\nabla \times \boldsymbol{H} = j\omega\epsilon \boldsymbol{E}$ yields the following equation for E_ϕ:

$$E_\phi = \frac{1}{j\omega\epsilon h_s}\left\{-\gamma H_\rho - \frac{\partial}{\partial\rho}(h_s H_{s1})\right\}. \tag{29}$$

Expanding this equation to first order in R^{-1}, we find that $E_{\phi 1}$ satisfies

$$E_{\phi 1} = \frac{-k}{\epsilon\omega}H_{\rho 1} + \frac{j}{\epsilon\omega}\frac{\partial}{\partial\rho}H_{s1}. \tag{30}$$

Through a utilization of (9) and (23), we obtain the solution for $E_{\phi 1}$ in final form, namely,

$$\begin{aligned} E_{\phi 1}(\rho,\phi) = -\sqrt{\frac{\mu}{\epsilon}}\,\frac{I\sin\phi}{\pi(b^2-a^2)}\Bigg\{ & \frac{k^2a^2b^2k_1}{(b^2-a^2)\rho^2} \\ & - \frac{a^2b^2}{4\rho^2}\ln(a/b) + \frac{k^2k_2}{(b^2-a^2)} \\ & + \frac{1}{4}f_1(\rho) + \frac{7}{32}k^2\rho^2(b^2-a^2) - \frac{k^2a^2b^2}{2(b^2-a^2)} \\ & \cdot\ln(a/b)f_1(\rho) - \frac{k^2\rho^2}{8}f_1(\rho)\Bigg\}. \end{aligned} \tag{31}$$

This expression, it should be noted, satisfies the boundary condition that the tangential component of the electric field must vanish at the surface of a perfect conductor, i.e., $E_{\phi 1} = 0$ at $\rho = a$ and $\rho = b$. With the construction of (31), we have completed the task of obtaining solutions for all field components from Maxwell's equations to first order in the inverse of the radius of curvature of the bent coaxial line.

V. Second-Order Axial Fields

To find the first nonvanishing correction to the propagation constant γ, it is necessary to proceed to the second order in R^{-1}. Firstly, second-order solutions for the propagating axial field components must be derived from the appropriate differential equations for H_s and E_s. These second-order solutions, denoted by H_{s2} and E_{s2}, are just the coefficients of the second-order terms in the expansions of H_s and E_s in inverse powers of R, i.e.,

$$H_s = H_{s0} + \frac{H_{s1}}{R} + \frac{H_{s2}}{R^2} + \cdots$$

$$E_s = E_{s0} + \frac{E_{s1}}{R} + \frac{E_{s2}}{R^2} + \cdots. \tag{32}$$

Secondly, H_{s2} and E_{s2} must be substituted into the differential equation governing the second-order solution for the axial field combination $G_{\pm,2}$, defined by

$$G_{\pm,2} = E_{s2} \pm j\sqrt{\frac{\mu}{\epsilon}}\,H_{s2}. \tag{33}$$

It is then a straightforward matter to solve for A_2, the coefficient of the second-order term in the expansion of γ^2, i.e.,

$$\gamma^2 = \gamma_0^2\left(1 + \frac{A_2}{R^2} + \cdots\right). \tag{34}$$

In writing this relationship, we have used the previously deduced fact that the first-order correction to γ^2 vanishes.

The partial differential equation from which H_{s2} is obtainable can be constructed by expanding (4) to second order in R^{-1} and, subsequently, retaining only those terms proportional to R^{-2}. Since terms containing the first-order field solutions H_{s1}, $H_{\rho 1}$, and $H_{\phi 1}$ appear in the resulting expression, it is then necessary to utilize the previously derived solutions found in (9), (23), and (25). As a result, we find that the differential equation governing H_{s2} becomes, after some rearrangement,

$$\nabla_t^2 H_{s2} = jkI\sin 2\phi\left\{\frac{a_1}{\rho^2} + a_2\rho^2 f_1(\rho) + a_3\rho^2 + a_4\right\} \tag{35}$$

where

$$a_1=\frac{1}{2\pi(b^2-a^2)}\left\{-\frac{4k^2a^2b^2k_1}{(b^2-a^2)}-2a^2b^2\ln(a/b)\right\}$$
$$a_2=\frac{3k^2}{4\pi(b^2-a^2)}$$
$$a_3=-\frac{11k^2}{16\pi}$$
$$a_4=\frac{1}{2\pi(b^2-a^2)}\{3(b^2-a^2)+k^2a^2b^2\ln(a/b)\}. \quad (36)$$

It can be shown that the solution for H_{s2} which satisfies the appropriate boundary conditions is given by

$$H_{s2}(\rho,\phi)=\frac{jkI\sin 2\phi}{\pi(b^2-a^2)}\left\{h_1\rho^2+h_2\rho^2f_2(\rho)+\frac{h_3}{\rho^2}+h_4\rho^4f_1(\rho)+h_5\rho^4+h_6\right\} \quad (37)$$

where f_2 is the following function of ρ:

$$f_2(\rho)=b^4\ln(\rho/b)-a^4\ln(\rho/a). \quad (38)$$

The various h_i appearing in (37) are constants defined by

$$h_1=\frac{\pi}{2(b^2+a^2)}\{b_5(a^4-b^4)+4b_2a^2b^2\ln(a/b)(g_1a^4-g_2b^4)+4b_3(a^6-b^6)+b_2(b^2-a^2)(g_3a^6-g_4b^6)\}$$
$$h_2=\frac{\pi b_5}{b^2+a^2}$$
$$h_3=\frac{\pi}{2(b^2+a^2)}\left\{a^4b^4(a^2-b^2)[4b_3-b_2(a^2-b^2)]+2a^4b^4\ln(a/b)\left[b_5+\frac{4b_2(a^2-b^2)}{k^2}\right]\right\}$$
$$h_4=\pi(b^2-a^2)b_2$$
$$h_5=\pi(b^2-a^2)b_3$$
$$h_6=\pi(b^2-a^2)b_4 \quad (39)$$

with

$$g_1=1-\frac{2}{k^2a^2} \qquad g_2=1-\frac{2}{k^2b^2}$$
$$g_3=1+\frac{4}{k^2a^2} \qquad g_4=1+\frac{4}{k^2b^2}$$
$$b_2=\frac{a_2}{12} \qquad b_3=\frac{a_3}{12}-\frac{a_2}{18}(b^2-a^2)$$
$$b_4=\frac{-a_1}{4} \qquad b_5=\frac{a_4}{4}. \quad (40)$$

The solution for H_{s2} given by (37) conforms to the boundary conditions that $\partial H_{s2}/\partial\rho=\cos\phi H_{s1}$ at $\rho=a$ and $\rho=b$, which are deducible from Maxwell's equations.

By following a procedure similar to that outlined above, the partial differential equation for the axial electric field to second order in R^{-1} can be deduced. The result, which follows from a consideration of the explicit analytical forms for the first-order field components E_{s1}, $E_{\rho 1}$, and $E_{\phi 1}$, is

$$\nabla_t^2E_{s2}=jk\eta I\left\{\cos 2\phi\left[\frac{c_1}{\rho^2}+c_2\rho^2f_1(\rho)+c_3\rho^2+c_4\right]+c_5+c_6\rho^2+c_7f_1(\rho)+c_8\rho^2f_1(\rho)\right\} \quad (41)$$

where η denotes the intrinsic impedance of the dielectric medium between the conductors of the coaxial line, i.e.,

$$\eta=\sqrt{\frac{\mu}{\epsilon}}. \quad (42)$$

The c_i coefficients appearing in (41) are defined as

$$c_1=\frac{1}{\pi(b^2-a^2)}\left\{\frac{2k^2a^2b^2k_1}{(b^2-a^2)}-a^2b^2\ln(a/b)\right\}$$
$$c_2=-\frac{3k^2}{4\pi(b^2-a^2)}$$
$$c_3=\frac{3k^2}{16\pi}$$
$$c_4=\frac{1}{\pi(b^2-a^2)}\left\{-\frac{3}{2}(b^2-a^2)+\frac{k^2a^2b^2}{2}\ln(a/b)\right\}$$
$$c_5=\frac{1}{\pi(b^2-a^2)}\left\{-\frac{2k^2k_2}{(b^2-a^2)}-\frac{3}{2}(b^2-a^2)+\frac{k^2a^2b^2}{2}\ln(a/b)\right\}$$
$$c_6=-\frac{k^2}{4\pi}$$
$$c_7=\frac{1}{\pi(b^2-a^2)}\left\{\frac{k^2a^2b^2}{(b^2-a^2)}\ln(a/b)-1\right\}$$
$$c_8=\frac{-k^2}{2\pi(b^2-a^2)}. \quad (43)$$

Once again following lengthy computations, we arrive at the following analytical solution for E_{s2}:

$$E_{s2}(\rho,\phi)=\frac{jk\eta I}{\pi(b^2-a^2)}\left\{\cos 2\phi\left[e_1\rho^2+e_2\rho^2f_2(\rho)+\frac{e_3}{\rho^2}+e_4\rho^4f_1(\rho)+e_5\rho^4+e_6\right]+\tilde{c}_5\frac{\rho^2}{4}+\tilde{c}_6\frac{\rho^4}{16}+\tilde{c}_7\frac{\rho^2}{4}f_1(\rho)+\tilde{c}_7\frac{\rho^2}{4}(a^2-b^2)+\tilde{c}_8\frac{\rho^4}{16}f_1(\rho)+\tilde{c}_8\frac{\rho^4}{32}(a^2-b^2)+\tilde{c}_9\ln\rho+\tilde{c}_{10}\right\}. \quad (44)$$

The various e_i found in this equation are constants defined by

$$e_1 = \frac{\pi}{(b^2+a^2)}\{d_2a^2b^2(a^4-b^4)\ln(a/b) + d_3(a^6-b^6) + d_4(a^2-b^2)\}$$

$$e_2 = \frac{\pi d_5}{b^2+a^2}$$

$$e_3 = \frac{\pi}{(b^2+a^2)}\{d_3a^4b^4(b^2-a^2) + d_4a^2b^2(a^2-b^2) - d_5a^4b^4\ln(a/b)\}$$

$$e_4 = \pi(b^2-a^2)\,d_2$$

$$e_5 = \pi(b^2-a^2)\,d_3$$

$$e_6 = \pi(b^2-a^2)\,d_4 \qquad (45)$$

where

$$d_2 = \frac{c_2}{12} \qquad d_3 = \frac{c_3}{12} - \frac{c_2}{18}(b^2-a^2)$$

$$d_4 = -\frac{c_1}{4} \qquad d_5 = \frac{c_4}{4}. \qquad (46)$$

With regard to the coefficients $\tilde{c}_5$ through $\tilde{c}_8$ appearing in (44), they are expressible in terms of the previously defined c_i by

$$\tilde{c}_i = \pi(b^2-a^2)c_i \qquad (47)$$

where $i = 5, 6, 7, 8$. Expressions for the remaining $\tilde{c}_i$ coefficients, $\tilde{c}_9$ and $\tilde{c}_{10}$, are

$$\tilde{c}_9 = \frac{(b^2-a^2)}{\ln(a/b)}\left\{\frac{7k^2}{64(b^2-a^2)}\left[b^6+a^6-a^2b^2(b^2+a^2)\right] - \frac{7}{32}k^2a^2b^2\ln(a/b) - \frac{1}{8}(b^2-a^2)\right\}$$

$$\tilde{c}_{10} = \frac{(a^2\ln b - b^2\ln a)}{(b^2-a^2)}\tilde{c}_9 - \frac{a^2b^2}{4}\ln(a/b)\tilde{c}_7. \qquad (48)$$

As with E_{s1}, E_{s2} satisfies the boundary conditions $E_{s2}(a,\phi) = E_{s2}(b,\phi) = 0$; i.e., it vanishes at the surfaces of the inner and outer conductors. With the development of (37) and (44), the solution of Maxwell's equations for the axial fields to second order in R^{-1} is complete.

VI. Second-Order Correction to Propagation Constant

The partial differential equation governing the second-order axial field combination $G_{\pm,2}$, defined by (33), is deducible by retaining only terms of second order in R^{-1} in the general differential equation for $G_\pm$, i.e., (14). The result, obtainable with the aid of (34) and the last expression in (2), is

$$-\rho\cos\phi\nabla_t^2G_{\pm,2} + \left(\frac{5}{2}\rho^2\cos^2\phi - \frac{A_2}{2}\right)\nabla_t^2G_{\pm,1} + 2k^2\rho^2\cos^2\phi G_{\pm,1}$$
$$= G_{\pm,1} - \left(\cos\phi\frac{\partial}{\partial\rho}G_{\pm,2} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}G_{\pm,2}\right) \pm j\left(\sin\phi\frac{\partial}{\partial\rho}G_{\pm,2} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}G_{\pm,2}\right)$$
$$\mp j2\rho\cos\phi\left(\sin\phi\frac{\partial}{\partial\rho}G_{\pm,1} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}G_{\pm,1}\right). \qquad (49)$$

Upon substitution of the derived analytical forms for the axial fields which comprise $G_{\pm,1}$ and $G_{\pm,2}$, specifically (9), (12), (37), and (44), equation (49) evolves into a multitude of terms. It turns out that the resulting equation is satisfied only if A_2 has the following functional form:

$$A_2 = -\frac{2}{\ln(a/b)}\left\{\frac{7k^2}{64(b^2-a^2)}\left[b^6+a^6-a^2b^2(b^2+a^2)\right] - \frac{7}{32}k^2a^2b^2\ln(a/b) - \frac{1}{8}(b^2-a^2)\right\} + \frac{2k^2a^2b^2}{(b^2-a^2)^2}\left\{\frac{7}{32}(b^2-a^2)^2 + \frac{a^2b^2}{2}\ln^2(a/b)\right\}. \qquad (50)$$

This expression then, as an examination of (34) reveals, constitutes the desired first nonvanishing correction to the propagation constant.

VII. Graphical Results

Plotted in Fig. 2 are curves depicting how the fields which exist in a straight section of coaxial line, namely E_ρ

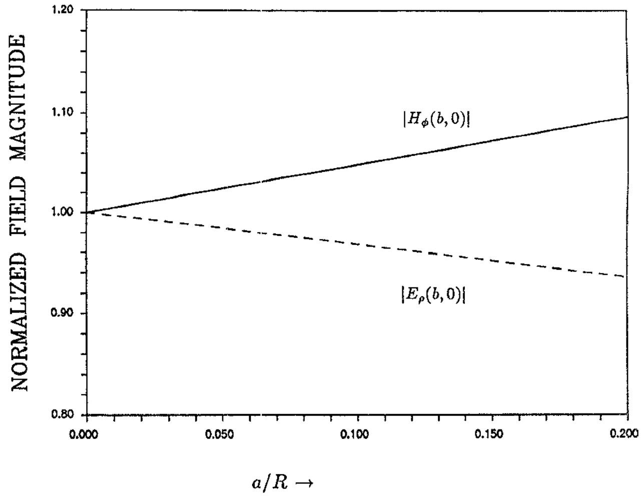

Fig. 2. Dependence of E_ρ and H_ϕ upon radius of curvature for 141 cable at 10 GHz.

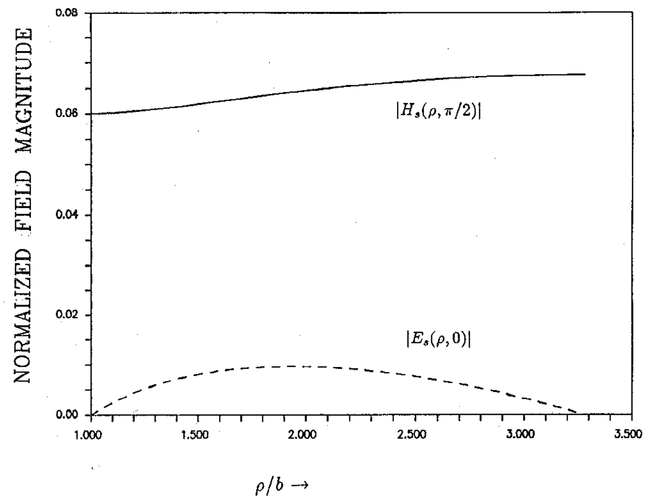

Fig. 3. Dependence of E_s and H_s upon radial coordinate for 141 cable at 10 GHz.

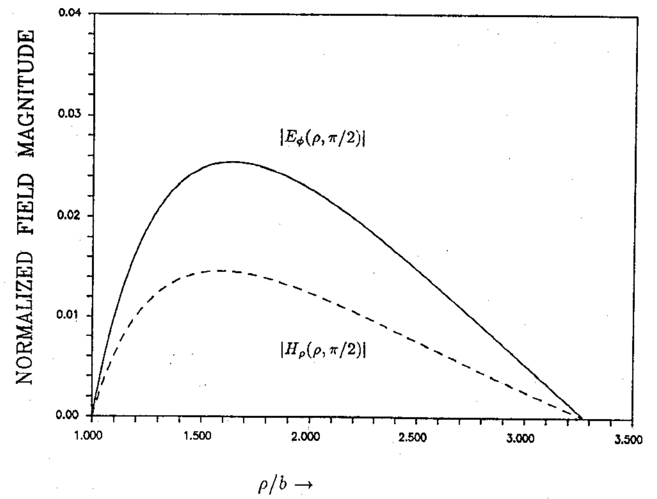

Fig. 4. Dependence of E_ϕ and H_ρ upon radial coordinate for 141 cable at 10 GHz.

and H_ϕ, change upon entering a bent section of coaxial line. The independent variable here is the ratio of the outer conductor (inner) radius to the transmission line radius of curvature, a/R. Hence, $a/R = 0$ represents the case of infinite radius of curvature, i.e., the straight coaxial line. The dimensions of the transmission line employed here are those which are appropriate for the commonly used 141 solid dielectric cable, in which Teflon serves as the dielectric. For this particular line, the outer conductor outer diameter is 0.141 in, the characteristic impedance is 50 Ω, and $a/b = 3.27$. With regard to the range of the variable a/R, it should be mentioned that mechanically a safe "bend" radius for the 141 cable corresponds to $a/R = 0.18$, while an extreme case in which the inner bend radius equals the cable radius corresponds to $a/R = 0.42$. The field magnitudes appearing in Fig. 2, deducible through the use of (1), (25), and (28), are evaluated at $\rho = b$ and $f = 10$ GHz and subsequently normalized to unity at $a/R = 0$. An examination of this figure reveals that $|E_\rho|$ decreases, whereas $|H_\phi|$ increases, upon entering a bent section of the coaxial line.

With regard to the field components that do not exist in a straight coaxial transmission line, these are plotted in Figs. 3 and 4 for the bent 141 solid dielectric cable. In particular, the (first-order) axial fields E_s and H_s, which are calculable from (9) and (12), appear in Fig. 3, while the transverse fields E_ϕ and H_ρ, calculable from (23) and (31), appear in Fig. 4. In these graphs, the independent variable is the normalized radial coordinate, ρ/b, while the radius of curvature is held fixed at $R = 0.147$ in (which corresponds to $a/R = 0.4$). The field magnitudes, which are evaluated at 10 GHz as in Fig. 2, are referenced to $E_{\rho 0}$ $(\rho = b)$ for the electric field components and to $H_{\phi 0}$ $(\rho = b)$ for the magnetic field components. It is seen from Figs. 3 and 4 that even in the case of the severe radius of curvature under consideration, the field components which exist only in a bent section of the 141 coaxial line amount to less than 7% of the straight-line unperturbed fields.

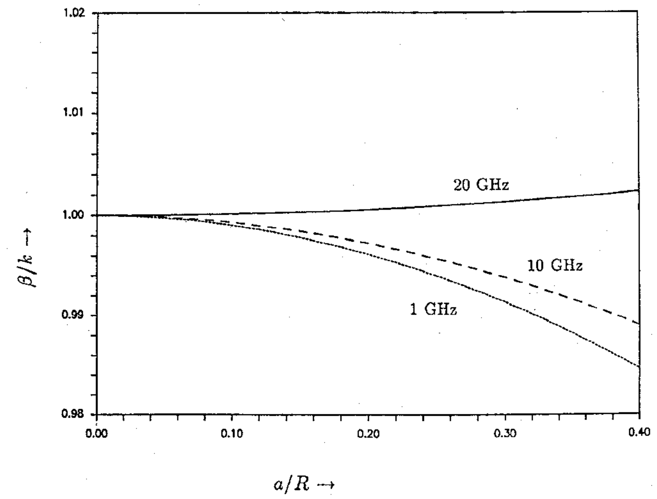

Fig. 5. Dependence of propagation constant upon radius of curvature for 141 cable.

Plotted in Fig. 5 are curves depicting the dependence of the normalized propagation constant upon the radius of curvature of a bent coaxial transmission line. Once again, a bent section of 141 solid dielectric cable is assumed to be the propagating medium for the perturbed TEM mode. The dependent variable plotted here is β/k, where $\beta = \gamma/j$, and as such is the ratio of the phase constant for the bent cable to that for the straight cable since A_2 is real. Three curves, each corresponding to one of the three commonly utilized operating frequencies of 1, 10, and 20 GHz for the 141 cable, appear in Fig. 5. An examination of this figure reveals that the propagation constant can increase (the 20 GHz curve does very slightly) or decrease when the propagating TEM mode enters a bend, depending upon the value of frequency. It is noted that results which are numerically quite close to those of Fig. 5 can be obtained for a 085 solid dielectric cable, which is also a commonly used coaxial line. This cable has an outer conductor outer diameter of 0.085 in and a characteristic impedance of 50 Ω.

VIII. Conclusions

A TEM mode propagating along a coaxial transmission line will cease to be purely TEM in character upon entering a bend, since there it will develop axial components of both the electric and magnetic fields. Moreover, an azimuthal electric field and a radial magnetic field, not present in the pure TEM mode, will arise in the bend. With regard to the existing azimuthal magnetic field and radial electric field, they will experience some change in magnitude in the bent section of line. In all cases, the alterations in the electromagnetic field structure will be inversely proportional to the radius of curvature of the bend, for small amounts of curvature. On the other hand, the propagation constant of the wave suffers a change in magnitude inversely proportional, for small curvature, to the square of the radius of curvature. In addition, this change in magnitude is dependent upon the values of the straight-line propagation constant and the inner and outer conductor radii.

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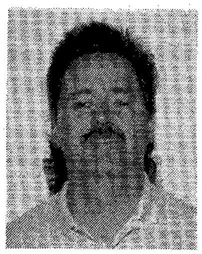

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